

# The Weyl character formula, the half-spin representations, and equal rank subgroups

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February 1, 2008

*Proc Natl. Acad. Sci. USA*, vol 95, pp. 8441-8442, July 1988

## Abstract

Let  $B$  be a reductive Lie subalgebra of a semi-simple Lie algebra of the same rank both over the complex numbers. To each finite dimensional irreducible representation  $V_\lambda$  of  $F$  we assign a multiplet of irreducible representations of  $B$  with  $m$  elements in each multiplet, where  $m$  is the index of the Weyl group of  $B$  in the Weyl group of  $F$ . We obtain a generalization of the Weyl character formula; our formula gives the character of  $V_\lambda$  as a quotient whose numerator is an alternating sum of the characters in the multiplet associated to  $V_\lambda$  and whose denominator is an alternating sum of the characters of the multiplet associated to the trivial representation of  $F$ .

UFIFT-HEP-98-21

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Spin(9) is the light cone little group of physical theories in 10+1 dimensions and can also be viewed as the little group of massive representations in 9+1 dimensions. Computations in N=1 supergravity in 10+1 dimensions led to the empirical discovery of families of triples of irreducible representations of Spin(9) with several remarkable properties: the dimension of one of the three is equal to the sum of the dimensions of the other two, and the second order Casimir, together with several of the higher order Casimirs take on the same value in all three representations (T. Pengpan and P. Ramond, unpublished data). The purpose of this note is to explain this phenomenon and place it in a general setting. In so doing, we obtain a formula relating virtual representations of Lie groups of the same rank which reduces to the Weyl character formula when the smaller group is a maximal torus.

So let  $B \subset F$  be two Lie algebras over the complex numbers with  $F$  semi-simple and  $B$  reductive and having the same rank. Choose a common Cartan subalgebra  $H$  with the roots of  $B$  being a subset of the roots of  $F$ . Thus the Weyl group  $W(B)$  of  $B$  is a subgroup of the Weyl group  $W(F)$  of  $F$ . Choose the positive roots consistently. Then the positive Weyl chamber  $\mathcal{W}_F$  of  $F$  is contained in the positive Weyl chamber  $\mathcal{W}_B$  of  $B$ . Let  $C \subset W(F)$  be those elements in  $W(F)$  which map  $\mathcal{W}_F$  into  $\mathcal{W}_B$ . So the cardinality of  $C$  is the index of  $W(B)$  in  $W(F)$  which we denote by  $\mathbf{m}$  and

$$\mathcal{W}_B = \bigcup_{w \in C} w\mathcal{W}_F,$$

while

$$W(F) = W(B) \cdot C.$$

(In the case of Spin(9), we take  $B = B_4$  and  $F = F_4$  in which case  $C$  consists of three elements.)

We let  $\rho_F$  denote one half the sum of the positive roots of  $F$  and  $\rho_B$  denote one half the sum of the positive roots of  $B$ . Let  $\lambda$  be a dominant weight of  $F$ . Let  $V_\lambda$  be the corresponding irreducible representation of  $F$  considered as a  $B$ -module by restriction. For each  $c \in C$ , let

$$c \bullet \lambda := c(\lambda + \rho_F) - \rho_B.$$

Then  $c \bullet \lambda$  is a dominant weight for  $B$ . We let  $U_{c \bullet \lambda}$  denote the irreducible representation of  $B$  with highest weight  $c \bullet \lambda$ . Finally, let  $P$  denote the orthogonal complement of  $B$  in  $F$  (under the Killing form of  $F$ ) so the adjoint representation of  $B$  on  $F$  gives an embedding of  $B$  into the orthogonal algebra  $\mathfrak{o}(P)$ . Since  $P$  is even dimensional,  $\mathfrak{o}(P)$  has two half-spin representations. Let  $S^\pm$  denote these half-spin representations considered as  $B$  modules. We claim that

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{c \in C} \text{sgn}(c) U_{c \bullet \lambda} \quad (1)$$

(for the appropriate choice of  $\pm$ ). Equation (1) is to be regarded as an equality in the ring of virtual representations of  $B$ .

Recall that  $\mathbf{m}$  denotes the index of  $W_B$  in  $W_F$ . The elements  $c \bullet \lambda$  as  $c$  ranges over  $c \in C$  are distinct and hence any  $\lambda \in \mathcal{W}_F$  defines a  $\mathbf{m}$ -multiplet  $\{c \bullet \lambda\}$ ,  $c \in C$  in  $\mathcal{W}_B$ . This generalizes the triplets which arise in the case where  $B = B_4$  and  $F = F_4$ . The multiplets may be abstractly characterized as follows: Using the Harish-Chandra isomorphism there is a natural injective homomorphism  $\eta : Z_F \rightarrow Z_B$  where  $Z_F$  and  $Z_B$  are respectively the centers of the enveloping algebras of  $F$  and  $B$ . Let  $Z_B^F \subset Z_B$  be the image of  $\eta$ . Let  $\mathcal{W}_B^*$  be the set of all dominant weights  $\nu \in \mathcal{W}_B$  such that  $\nu + \rho_B$  is a regular integral weight for  $F$ . Define an equivalence relation in  $\mathcal{W}_B^*$  by putting  $\mu \sim \nu$  if the infinitesimal characters of  $U_\mu$  and  $U_\nu$  agree on  $Z_B^F$ .

**Proposition** *The equivalence classes all have cardinality  $\mathbf{m}$  and each such class consists of the multiplets appearing on the right side of (1) for a suitable  $\lambda$ .*

Note that if  $B$  is simple, as in the case  $B = B_4$  then the quadratic Casimir is in  $Z_B^F$  and hence takes the same values for the representations whose highest weight are in a multiplet.

As far as we know the formula (1) is not in the literature although Wilfred Schmid informs us that he was aware of the result. The equation (1) may be rewritten as in (6) below where it takes the form of a branching law. The proof of (1) combines the use of the section  $C$  with Weyl's character formula. A recent citation for the idea of using a section of Weyl groups in combination with the character formula to obtain a branching law appears in Section 8.3.4 in ref[1]. There the branching is for the pair  $C_n, C_{n+1}$ .

Because the two representations occurring on the left hand side of (1) have the same dimension, we conclude that

$$\sum_{c \in C} \text{sgn}(c) \dim(U_{c \bullet \lambda}) = 0. \quad (2)$$

To prove (1) we examine the Weyl character formula for  $F$  which says that the character of  $V_\lambda$  is given by

$$\text{ch}(V_\lambda) = \frac{A_{\lambda + \rho_F}^F}{A_\rho^F}$$

where

$$A_{\lambda + \rho_F}^F = \sum_{w \in W(F)} \text{sgn}(w) e^{w(\lambda + \rho_F)}$$

and where the Weyl denominator has the two expressions

$$A_{\rho_F}^F = \sum_{w \in W(F)} \text{sgn}(w) e^{w(\rho_F)} = \prod \left( e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}} \right)$$

where the product is over all positive roots of  $F$ .

We do the summation that occurs in  $A_{\lambda+\rho_F}^F$  as a double sum (over  $W(F) = W(B) \cdot C$ ) to get

$$A_{\lambda+\rho_F}^F = \sum_{c \in C} \text{sgn}(c) A_{c \bullet \lambda + \rho_B}^B \quad (3)$$

where  $A_{c \bullet \lambda + \rho_B}^B$  is the numerator of the Weyl character formula of  $B$  associated to  $U_{c \bullet \lambda}$ . We may apply this to the Weyl denominator as well, taking  $\lambda = 0$ . This gives

$$\text{ch}(V_\lambda) = \frac{\sum_{c \in C} \text{sgn}(c) A_{c \bullet \lambda + \rho_B}^B}{\sum_{c \in C} \text{sgn}(c) A_{c \bullet 0 + \rho_B}^B}. \quad (4)$$

We may also rewrite the product expression for the Weyl denominator for  $F$  as follows: Write the product over all positive roots of  $F$  as the product over all positive roots of  $B$  times the product over all positive missing roots:

$$A_{\rho_F}^F = A_{\rho_B}^B D$$

with

$$\Delta := \prod_{\psi \in \Phi^+(F/B)} \left( e^{\frac{\psi}{2}} - e^{-\frac{\psi}{2}} \right)$$

where  $\Phi^+(F/B)$  denotes the set of positive roots of  $F$  that are not roots of  $B$ . We thus obtain

$$\text{ch}(V_\lambda) = \frac{1}{\Delta} \sum_{c \in C} \text{sgn}(c) \text{ch}(U_{c \bullet \lambda}). \quad (5)$$

Multiplying out the product in  $\Delta$  we see that

$$\Delta = \text{ch } S^+ - \text{ch } S^-$$

and multiplying (5) by  $\Delta$  proves (1).

If we divide the numerator and denominator of (4) by  $A_{\rho_B}^B$ , the Weyl denominator for  $B$ , we obtain

$$\text{ch}(V_\lambda) = \frac{\sum_{c \in C} \text{sgn}(c) \text{ch } U_{c \bullet \lambda}}{\sum_{c \in C} \text{sgn}(c) \text{ch } U_{c \bullet 0}} \quad (6)$$

and comparing (6) with (5), or simply taking  $\lambda = 0$  in (1), we see that

$$\Delta = \sum_{c \in C} \text{sgn}(c) \text{ch } U_{c \bullet 0}. \quad (7)$$

Notice that  $\Delta$  vanishes on the hyperplanes determined by the  $\psi \in \Phi^+(F/B)$  rather than on all the root hyperplanes as in the Weyl character formula. So the right hand side of (5) or equivalently of (6) makes sense on the complement of the hyperplanes corresponding to the  $\psi \in \Phi^+(F/B)$ .

If we take the subgroup  $B$  to be the maximal torus itself, equations (1), (4), (5), or (6) are just restatements of the Weyl character formula. If we take  $B = B_4$  and  $F = F_4$ , there are exactly three terms on the right of (1), since the Weyl group of  $B_4$  has index three in the Weyl group of  $F_4$ . Thus, in this case, equation (2) says that the dimension of the representation occurring with a minus sign on the right of (2) is equal to the sum of the dimensions of the other two. In the case  $B = D_8$  and  $F = E_8$  there are 135 terms on the right of (1).

If we take  $B = D_n$  and  $F = B_n$  we obtain a formula for the character of a representation of  $o(2n+1)$  in terms of characters of  $o(2n)$ ; as the index of the Weyl groups is two, the right hand side of (4) contains two terms in the numerator and in the denominator. Explicitly, if we make the usual choice of Cartan subalgebra and positive roots,  $\epsilon_i \pm \epsilon_j$  ( $1 \leq i < j \leq n$ ) for  $D_n$  and these together with  $\epsilon_i$ , ( $1 \leq i \leq n$ ) for  $B_n$ , the interior of positive Weyl chamber  $\mathcal{W}_F$  for  $B_n$  consists of all  $x = (x_1, \dots, x_n) = x_1\epsilon_1 + \dots + x_n\epsilon_n$  with

$$x_1 > x_2 > \dots > x_n > 0$$

while the interior of positive Weyl chamber  $\mathcal{W}_B$  of  $D_n$  consists of all  $x$  satisfying

$$x_1 > x_2 > \dots > x_{n-1} > |x_n|.$$

Thus  $C$  consists of the identity  $e$  and the reflection  $s$ , which changes the sign of the last coordinate. Here

$$\rho_F = \left( \frac{2n-1}{2}, \frac{2n-3}{2}, \dots, \frac{1}{2} \right)$$

whereas

$$\rho_B = (n-1, n-2, \dots, 1, 0).$$

Thus for  $\lambda = (\lambda_1, \dots, \lambda_n)$  we have

$$e \bullet \lambda = (\lambda_1 + \frac{1}{2}, \dots, \lambda_n + \frac{1}{2}) \quad \text{and} \quad s \bullet \lambda = (\lambda_1 + \frac{1}{2}, \dots, \lambda_{n-1} + \frac{1}{2}, -\lambda_n - \frac{1}{2}).$$

Equation (4) becomes

$$\text{ch} V_\lambda = \frac{\text{ch} U_{e \bullet \lambda} - \text{ch} U_{s \bullet \lambda}}{U_{e \bullet 0} - U_{s \bullet 0}}. \quad (8)$$

Notice that in this special case  $S^+ = U_{e \bullet 0}$  and  $S^- = U_{s \bullet 0}$  and are the actual half spin representations of  $o(2n)$ . (For  $n = 1$  (8) reduces to the formula for the summation of a geometrical sum.)

## References

- [1] Goodman R. and Wallach, N. *Representations and Invariants of the Classical Groups*, Cambridge University Press 1998.